An Innovative Decision Support Approach to Hedge Path Dependent Risk in Option Market

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Abstract. We develop an innovative decision support approach to hedge the path dependent risk of underlying asset price using options. Nowadays, traditional risk measures describe the risk of final payoff of assets. We define how much time averagely the price will be under a certain level before a certain date as Time Risk Measure. We develop a new decision support procedure to compute and hedge Time Risk Measure. Under Geometric Brownian Motion, the inverse Laplace expression of Time Risk Measure can be solved. This can be considered as a Geometric Brownian Motion version of Levy's Arcsine Law.

Keywords: Pathwise Risk, Decision Support, Levy's Arcsine Law, Time Risk Measure

1. Introduction

This paper makes two contributions to decision support procedure in finance: (1) decision support to gauge a new risk measure of asset price path dependent behavior and (2) decision support to hedge the new risk measure using a new synthetic exotic American derivative.

We define the expected length of time when the forward price of stock will be below a certain level (more precisely, expected value of the Lebesgue measure of set of all time t, such that at time t the forward price of stock is below a certain level) as Time Risk Measure. Nowadays the usual risk measure of asset price is concerned by terminal value, i.e. the final distribution of asset price a month later. But very limited study has been conducted on the path dependent behavior of asset price(Carr and Wu, 2005 and 2009). Imagining that the equity holder and creditor of a company may have some tolerance about that the stock price stays below a certain level for a short period of time, they would like to measure the average length of time when the stock price will be under some dangerous level. So the time risk measure can be employed to gauge this type of risk in both theoretical and implied way. Either we can calculate the time risk measure assuming that the forward price follows a certain stochastic process, or we can calculate the implied time risk measure from option market data.

The demonstration in Figure 1 shows the definition of Time Risk Measure. In the first panel, the line shows one realization of future process of an asset. In this example, the future process is generated by Geometric Brownian Motion with sigma of 0.3 and start point at 1. The horizontal length of shadow area shows Time Risk Measure of this realization with level of 0.9. The second panel shows 5 realizations of future processes. We estimate theoretical Time Risk Measure of an asset by taking expectation of Time Risk Measure of all possible realizations in the future.

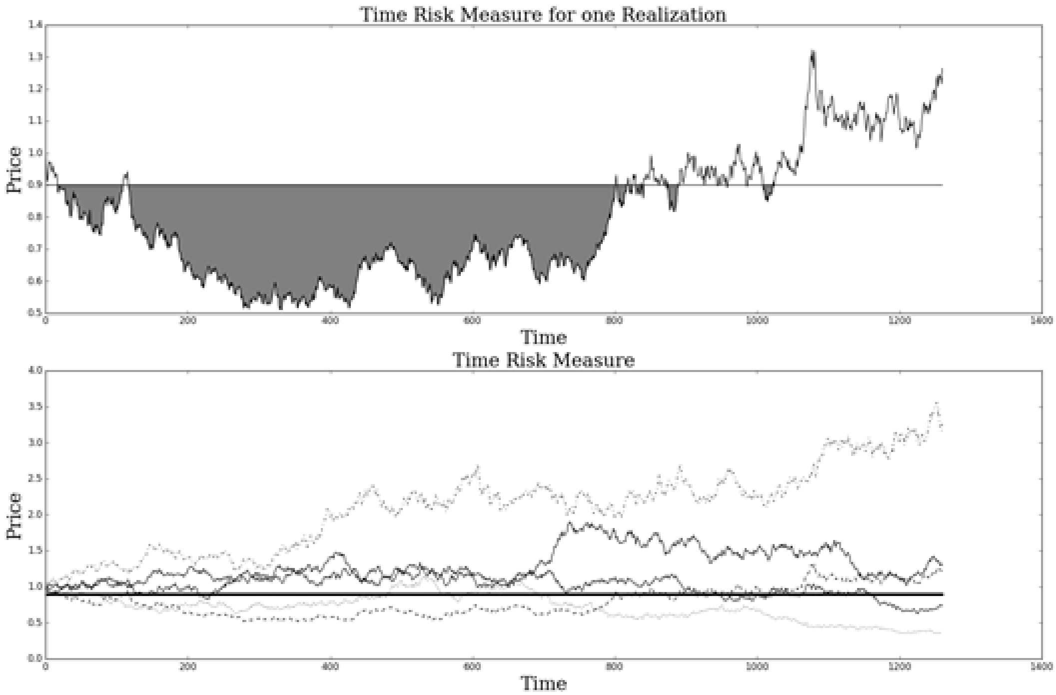


Figure 1 Demonstration of Time Risk Measure

The first panel shows the example of Time Risk Measure for one Realization, which is the length of time when the future process will be below the certain level. In this example, the line shows one realization of future process generated by Geometric Brownian Motion with sigma of 0.3 and start point at 1. The horizontal length of shadow area shows Time Risk Measure with level of 0.9. The second panel shows 5 realizations of Geometric Brownian Motion with sigma of 0.3 and start point at 1. The Time Risk Measure is the expected length of time when the future process will be below the certain level.

Also, the time risk measure itself is a payoff of an exotic American derivatives, can be replicated by a superposition of European options, and we name it Time Variance Swap. This makes the decision support procedure practical in option markets. The Time variance swap to the time risk measure is just like the variance swap to the VIX index. We construct the Time Variance Swaps with SPX options in CBOE and research the properties of Time Variance Swaps.

We find that implied time risk measure of SPX Index is always higher than the theoretical value from the time risk measure formula for Black-Scholes assumptions. The premium could be from two aspects: The first aspect is the volatility skew caused by that the forward process does not follow exactly Geometric Brownian Motion, and has left heavy tailed distributions or jumps. We can verify this by checking both implied volatility skew of the option and the implied volatility skew of the Time Variance Swap. The second is the investors who hedges the time risk measure would like to higher price, so the time risk measure beta gives a negative return like volatility beta does.

2. Methodology

According to CBOE (2009)，the formula of VIX Index with maturity T is a numerical integral of out-of-the-money call and put options.

$$VIX^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} O_T(K_i)$$

Where K_i is the i-th strike price, and R is the risk-free rate. $O_t(K,T)$ is the out-of-the-money option with strike K_i and maturity T. Demeterfi et al. (1999) introduces the derivation of the formula under assumptions of Black and Scholes (1973). Britten - Jones and Neuberger (2000) provide another interesting proof of VIX index under diffusion process assumption. Carr and Wu (2005) introduces how the VIX evolves，from the old VXO version to the modern VIX version. Carr and Wu (2009) discusses the derivation of the VIX formula in a broader assumptions, an exponential Lévy process. This assumption is a very general assumption, which nests Merton model(Merton, 1976), Variance Gamma model (Madan et al., 1998), Inverse Gaussian model (Rydberg, 1997) and etc.(Schilling, 2004)

We start with a risk-neutral probability measure $\mathbb{Q}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ such that the forward price follows the equation:

$$\mathrm{dF_t} = F_{t-}\sigma_{t-}dW_t + \int_{\mathbb{R}^0} F_{t-}(e^x - 1)[\mu(dx,dt) - v_t(x)dxdt] \tag{1}$$

Where W_t is a Q-standard Brownian motion, $\mathbb{R}^0$ denotes the real line excluding zero, F_{t-}denotes the futures price just prior to any jump at time t , and the Lévy measure $\mu(dx,dt)$ realizes to a nonzero value for a given x if and only if the futures price jumps from F_{t-} to $F_t = F_{t-}\mathrm{e}^{\mathrm{x}}$ at time t. The process $v_t(x)$ compensates the jump process, so that the last term in Equation is the increment of a Q-pure jump martingale.

Under the specification in Equation (1), the quadratic variation on the futures return from time t to T is

$$V_{t,T} \equiv \int_t^T \sigma_{s-}^2 ds + \int_t^T \int_{\mathbb{R}^0} x^2 \ \mu(dx,ds) \tag{2}$$

And the VIX index is defined as the conditional risk-neutral expectation of the annualized return variance over the next 30 calendar days:

$$\mathrm{VIX}^2 = E_t^{\mathbb{Q}}[\frac{1}{T-t}V_{t,T}] \tag{3}$$

To understand the replication strategy and appreciate the economic underpinnings of the new VIX, we follow Carr and Wu [2004] in decomposing the realized return variance into three components:

$$V_{t,T} = 2[\int_0^{F_t} \frac{1}{K^2}(K - S_T)^+ dK + \int_{F_t}^{\infty} \frac{1}{K^2}(S_T - K)^+ \, dK] + 2\int_t^T \left[\frac{1}{F_{s-}} - \frac{1}{F_t}\right] dF_s - 2\int_t^T \int_{\mathbb{R}^0} \left[e^x - 1 - x - \frac{x^2}{2}\right] \mu(dx,ds) \tag{4}$$

Where $\mathrm{S_t}$ denotes the time-t spot index level. The first term can be represented as the superposition of out of the money (in terms of future price level) options, the second term can be replicated by a dynamical trading strategy holding $2\mathrm{e}^{\mathrm{rt}}\left[\frac{1}{F_{s-}} - \frac{1}{F_t}\right]$futures at time s, and the third term is a higher order error caused by jump.

We combine (3) and (4) to get the calculation formula for VIX.

$$VIX^2 = \frac{2}{T-t}e^{r(T-t)}\int_0^{\infty} \frac{O_t(K,T)}{K^2} dK - \frac{2}{T-t}\int_t^T \int_{\mathbb{R}^0} \left[e^x - 1 - x - \frac{x^2}{2}\right] v_s(x)dxds \approx \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} O_t(K_i) \tag{5}$$

Where $O_t(K,T)$ is the out-of-the-money option price with strike K and maturity T. Carr and Wu (2009) have discussed two approximation error for VIX index. The first approximation error is the discretization error when the integral is estimated by discrete sum of real world options. The second approximation error is the error introduced by jump.

Theorem 1

$$\frac{2}{T-t}e^{r(T-t)}\int_0^{K_{min}} \frac{1}{K^2} O_t(K,T)dK = \frac{2}{T-t}E_t\left[\frac{1}{2}\int_t^T 1_{F_{s-}<K_{min}} \sigma_{s-}^2 ds\right] - \frac{2}{T-t}\int_t^T \int_{\mathbb{R}^0} [f(F_{s-}e^x) - f(F_{s-}) - f'(F_{s-})F_{s-}(e^x - 1)] v_s(x)dxds \tag{6}$$

We call the length of time when the future price will averagely be below K_0, $\int_t^T 1_{F_{s-}<K_0}\sigma_{s-}^2 ds$, as the Time Risk Measure with level of K_0 because it is a new type risk measure which describe intermediate behavior of asset price. For example, a financial institute may not only be concerned by the final payoff of an asset after a period of time, but also by its path dependent behavioral during the period. It is because the debt holders and the equity holders may have a tolerance that the asset price can be below a certain level, but not being too long. The Time Risk Measure can perfectly measure this type of risk. Also we define Time Risk Measure divided by total time period as Percentage Time Risk Measure.

Corollary 1

If the future process follows a driftless Geometric Brownian Motion:

$$dF_t = F_t \sigma_t dW_t$$

The following equality exists:

$$\frac{2}{T-t} e^{r(T-t)} \int_0^{K_{min}} \frac{1}{K^2} O_t(K,T) dK$$

$$= \frac{1}{T-t} \sigma^2 \ E_t \left[\int_t^T 1_{F_s < K_{min}} ds \right]$$

$$\equiv \frac{1}{T-t} \sigma^2 \times Time \ \ Risk \ \ Measure$$

(7)

Also we expand the Theorem 1 and its corollaries into call option version, and also we can find that correspond version of Time Variance Swap and Time Risk Measure with similar methods.

Theorem 2

$$\frac{2}{T-t} e^{r(T-t)} \int_{K_{max}}^{\infty} \frac{1}{K^2} O_t(K,T) dK$$

$$= \frac{2}{T-t} E_t \left[\frac{1}{2} \int_t^T 1_{F_s - > K_{max}} \sigma_{s-}^2 ds \right]$$

$$- \frac{2}{T-t} E_t \ \int_t^T \int_{\mathbb{R}^0} [f(F_{s-} e^x) - f(F_{s-})$$

$$- f'(F_{s-}) F_{s-} (e^x - 1)] v_s(x) dx ds$$

(8)

Corollary 2

If the future process follows a driftless Geometric Brownian Motion:

$$dF_t = F_t \sigma_t dW_t$$

The following equality exists:

$$\frac{2}{T-t} e^{r(T-t)} \int_{K_{max}}^{\infty} \frac{1}{K^2} O_t(K,T) dK$$

$$= \frac{1}{T-t} \sigma^2 E_t \left[\int_t^T 1_{F_s > K_{min}} ds \right]$$

(9)

The Corollary 1 and Corollary 2 shows that the Time Variance Swap can be expressed as the squared volatility times the Time Risk Measure. But if we want to get the explicit (or almost explicit) expression of Time Risk Measure $E_t \left[\int_t^T 1_{F_s > K_{min}} ds \right]$ for Geometric Brownian Motion, we need to use some version of Feynman-Kac Connection Formula (Pham, 2000). The similar problem for Brownian Motion is solved by Lévy (1939), and Lévy shows that $E_t \left[\int_t^T 1_{B_s > 0} ds \right]$equals to zero, and $\int_t^T 1_{B_s > 0} ds$ follows an arcsine distribution. The Theorem 3 show that analytical expression of Laplace Transform of Time Risk Measure $E_t \left[\int_t^T 1_{F_s < K_{min}} ds \right]$.

Theorem3 *(Lévy's Law for Geometric Brownian Motion)*

If the future process follows a driftless Geometric Brownian Motion:

$dF_t = F_t \sigma_t dW_t$ *with* $F_0 = x$

Let $u^{(k)}(t,x) = E[\int_{T-t}^T f^{(k)}(F_s) ds]$ *with* $f^{(k)}(y) = 1_{y<k}$ *and* $u^{(k)}(0,x) = 0$

We can find:

$$u^{(K_{min})}(T,x) = \int_0^T 1_{F_s < K_{min}} ds$$

(10)

The Laplace transform of $u^{(k)}(t,x)$*with respect to time,* $\hat{u}^{(k)}(\alpha,x)$ *have the explicit expression:*

$$\hat{u}^{(k)}(\alpha,x)$$

$$= \begin{cases} \frac{1}{\alpha^2} \left(\frac{k}{x}\right)^{-\frac{1-\sqrt{1+\frac{8}{\sigma^2}\alpha}}{2}} \frac{1+\sqrt{1+\frac{8}{\sigma^2}\alpha}}{2\sqrt{1+\frac{8}{\sigma^2}\alpha}} & (x > k) \\ \frac{1}{\alpha^2} + \frac{1}{\alpha^2} \left(\frac{k}{x}\right)^{-\frac{1+\sqrt{1+\frac{8}{\sigma^2}\alpha}}{2}} \frac{1-\sqrt{1+\frac{8}{\sigma^2}\alpha}}{2\sqrt{1+\frac{8}{\sigma^2}\alpha}} & (x < k) \end{cases}$$

(11)

Proof：See Appendix

The inverse Laplace transform of (11) has no easy analytical expression, but the inverse Laplace transform can be easily calculated by the method of Rydberg (1997). The numerical method calculates the inverse Laplace transform very fast, which is million times faster than Monte Carlo method as we will see in the next section. Also, the numerical method enables us to calculate the inverse function of Time Risk Measure, which the Monte Carlo is not able to do. The following corollary shows that if the level k is chosen to be the current future level x, the analytical expression of Time Risk Measure can be found.

3. Data

The data is End-of-Day option market data US CBOE. The SPX Option data comprises monthly and weekly contracts from Feb 2 2002 to Apr 22 2016, and the SPX Option data do not include Wednesday contracts and seasonal contracts. In this paper, we focus on the contract with the maturity nearest to 30 calendar days. The mid quote is used in all the following research. The data points without any valid bid price in a day are removed from sample.

As suggested by Coval and Shumway (2001), we use the normal return rather than log return. Because options always shift from out-of-the-money to in-the-money or from in-the-money to out-of-the-money, the value of option often change from a large value to near zero, or from near zero to a large value. The log return could be problematic in these situations. For example, when option value goes to near zero, log return could be negative infinity, but normal could be -100%. Therefore, in our study, we choose normal return rather than log return.

4. Empirical Results

First, we construct Time Variance Swaps with level of 100%, 97.5%, 95%, 90%, 85% and 80% and calculate the Implied Percentage Time Risk Measure by Theorem 3. The result is shown in Figure 2. The implied Percentage Time Risk Measure skyrocket when the market panics in 2008, 2010, 2011 and 2015. Also, we find that the Percentage Time Risk Measure with lower level will change more during the market panic. In Table 1, we see the medium of Percentage Time Risk Measures with level of 100%, 97.5%, 95%, 90%, 85% and 80% are repectfully 65.61%, 39.19%, 24.34%, 10.81%, 5.17%, 2.41%. This results are higher than the theoretical values assuming volatility is 15. This means market participants trade the Time Risk Measures higher than the theoritical price.

We can also find this phenominon in the realized price of Time Variance Swaps. In Figure 3, the realized prices of Time Variance Swaps are sightly higher than the theoritical prices in all levels of 100%, 97.5%, 95%, 90%, 85% and 80%. There are two reasons: (1) Like Variance Swap, Time Variance Swap also has the premium that the buyers who wants to hedge the Time Risk Measure and the volatility risk need to pay higher and the sellers of the derivative acquire more to tolerance the risks. (2) Time Variance Swap has also implied volatility skew to price in the tail risk just as option does.

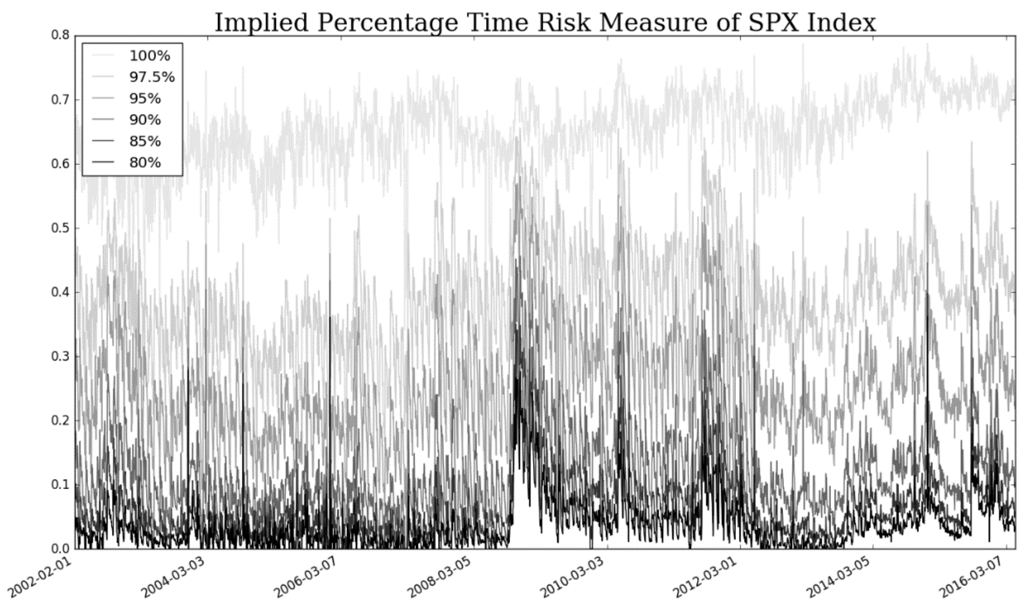


Figure 2 Implied Percentage Time Risk Measure of SPX Index
This figure shows Implied Percentage Time Risk Measure of SPX Index calculated by Theorem 3.

Table 1 Statistics of Implied Percentage Time Risk Measure of SPX Index

This table shows statistics of Implied Percentage Time Risk Measure of SPX Index calculated by Theorem 3.

Implied Percentage Time Risk Measure of SPX Index						
	100%	97.50%	95%	90%	85%	80%
Mean	65.00%	39.58%	25.70%	12.48%	6.57%	3.57%
Medium	65.61%	39.19%	24.34%	10.81%	5.17%	2.41%
Stdev	6.16%	8.47%	8.80%	7.24%	5.76%	4.69%
Variance	0.0038	0.0072	0.0078	0.0052	0.0033	0.0022
Kurtosis	10.8911	0.7078	2.0013	8.9690	25.1620	58.3978
Skewness	-1.8774	0.2301	0.9577	2.1714	3.5913	5.6422

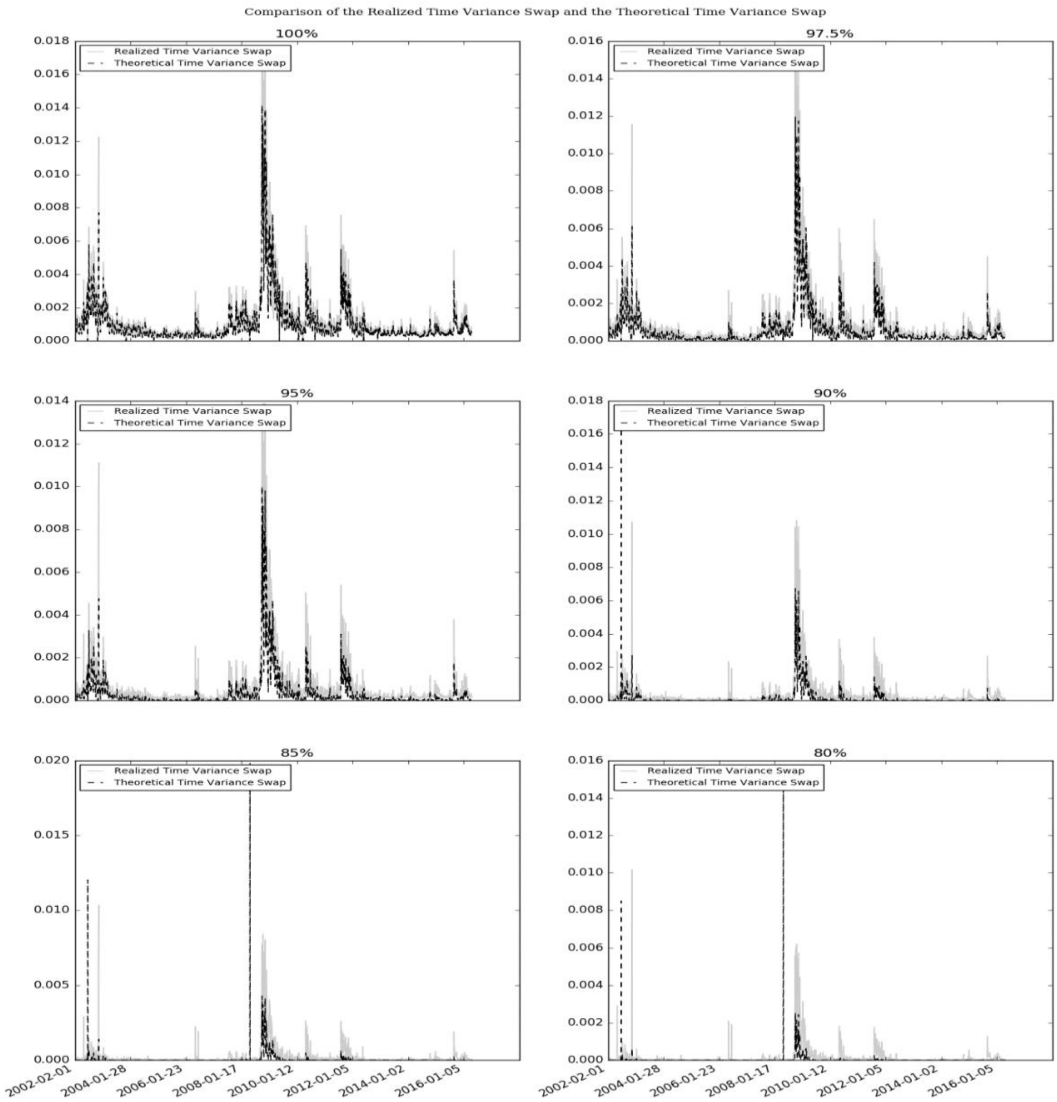


Figure 3 Comparison of the Realized Time Variance Swap and the Theoretical Time Variance Swap

The four panels compare respectively realized Time Variance Swap price and the theoretical Time Variance Swap Price by Theorem 3 with level of 100%, 97.5%, 95% and 90% of current future price. We find the market Time Variance Swap prices are higher than the theoretical Time Variance Swap prices in all four panels.

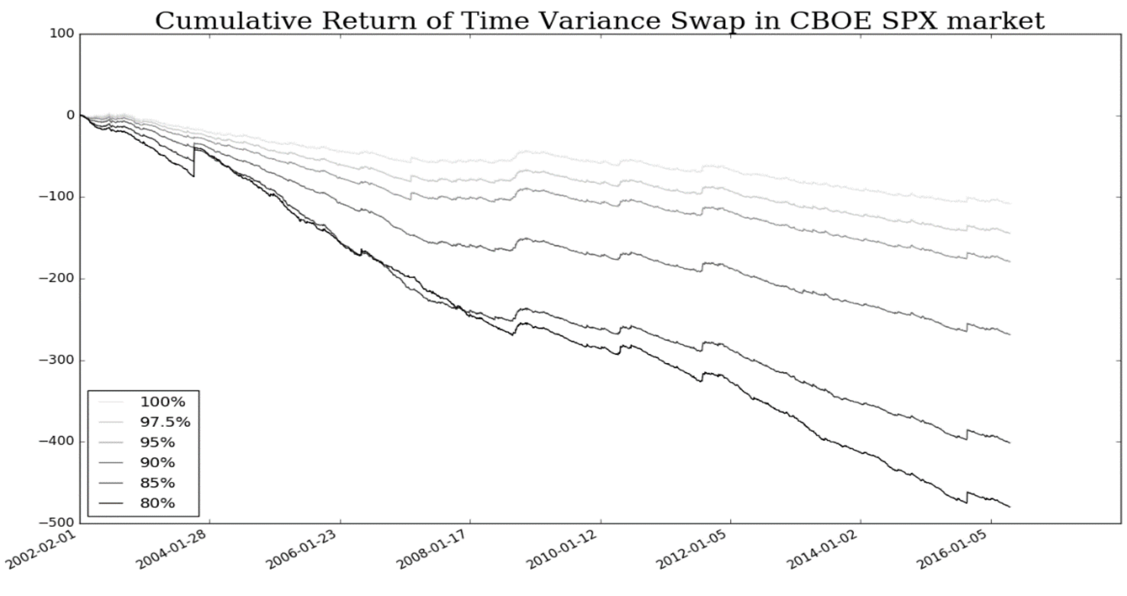
The first reason can be shown by the negative average of returns of Time Variance Swaps. In Table 2. The daily returns of Time Variance Swaps with all level of 100%, 97.5%, 95%, 90%, 85% and 80%, have negative mean and negative medium. The mean and medium of return of Time Variance Swap with lower level become more negative. We also see positive skewness of returns of Time Variance Swaps because that Time Variance Swaps usually have negative return but during market crushes they have very high returns. In Figure 4, we plot the arithematic cumulative returns Time Variance Swaps. We see they decreases smoothly except some spikes when market crushes happen. We see the lower level the Time Variance Swap is with, the cumulative return the Time Variance Swap has. This pheonominon that the options (or variance) are overpriced because investors have negative preference of volatility increasing, can be found in many literatures:

Coval and Shumway (2001) find that zero-beta at-the-money straddle positions produce average losses of approximately three percent per week. Jackwerth (2000) test the strategies which involves selling the put options with moneyness of 0.95 and 1.00. The number of puts sold is chosen such that the betas of the options strategies are about 1. Both of the strategies outperform purchasing the market index. Carr and Wu (2009) propose to use the difference between the realized variance and the variance swap rate to define the variance risk premium. They find that the variance risk premiums are strongly negative for the S&P and Dow indexes. Bollerslev et al. (2009)use variance risk premium as a market pricing factor, and find that variance risk premium is powerful in explaining the time-series variation in post-1990 aggregate stockmarket returns. Cremers et al. (2015) examine the performance of the pure vega portfolio and pure gamma portfolio by buying and short selling the near-term and long-term beta-zero straddles in S&P 500 futures options. Both of the portfolios have negative historical mean.

Table 2 Statistics of Daily Returns of Time Variance Swap in CBOE SPX market

The daily return is calculated by holding the Time Variance Swap with certain level (100%, 97.5%, 95%, 90%, 85% and 80% of current future price) for one day and rebalancing at the close on each day. The return is not log return, but normal return.

Return	TimeVarianceSwap100	TimeVarianceSwap97.5	TimeVarianceSwap95	TimeVarianceSwap90	TimeVarianceSwap85	TimeVarianceSwap80
Mean	-0.03046	-0.04058	-0.05033	-0.07563	-0.11448	-0.15511
Median	-0.09187	-0.11119	-0.12238	-0.1429	-0.16813	-0.2
Stdev	0.359431	0.38556	0.406369	0.424324	0.54044	0.862928
Skewness	2.742866	3.491815	4.365185	4.524768	10.69604	26.07735
Kurtosis	23.35196	35.17193	51.20367	45.77026	262.1318	1055.414



N	3564	3564	3563	3554	3507	3095

Figure 4 Cumulative Return of Time Variance Swap in CBOE SPX market

The daily return is calculated by holding the Time Variance Swap with certain level (100%, 97.5%, 95%, 90%, 85% and 80% of current future price) for one day and rebalancing at the close on each day. At the rebalancing, the portfolio will hold the same amount of value each day. Therefore, the cumulative return is the arithmetic sum of daily return of Time Variance Swap.

The second reason is that there is implied volatility skew in Time Variance Swaps. In other words, the Geometric Brownian Motion is not a perfect assumption for underlying asset in option pricing. In Figure 5, we can see similar pattern shared by implied volatility skew in both Time Variance Swaps and put options. We see the "smirk" pattern that the Time Variance Swap with lower level has a higher volatility. As Jackwerth (2000) found, typically after 1987 crash, the option traders have priced the tail risk of stock on options. Therefore, on one hand, we can use implied volatility skew as a pricing tool to overcome the shortage of Geometric Brownian Motion. On the other hand, we can use more sophisticated assumptions, such as Variance Gamma (Madan et al., 1998), Inverse Gaussian (Rydberg, 1997) and etc., to price the Time Variance Swaps.

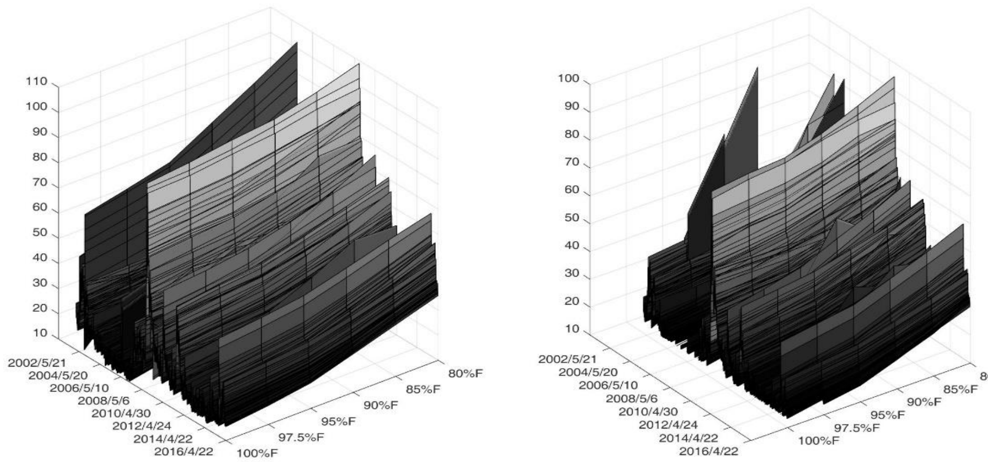


Figure 5 The Percentage Time Risk Measure Skew and Black-Scholes volatility Skew

The left panel is the Percentage Time Risk Measure Surface and the right panel is Black-Scholes volatility surface. Just like the Black-Scholes volatility surface, the Percentage Time Risk Measure Surface is the implied Percentage Time Risk Measure with different levels.

5. Conclusion

The study has two significances to decision support in option markets. First, we help investors to gauge a new type of path dependent risk, which we term Time Risk Measure. It describes the average length of time when the asset price will stay below some level during a period of time. So we are able to manage not only the risk of terminal asset price but also the path dependent risk of the asset price. Second, we provide the new synthetic derivatives, Time Variance Swaps, to enable investors to hedge the new type of risk.

In this study, we also have other important findings. These include the expression of integration of out-of-the-money options under assumption of exponential Lévy process. We have derived the almost analytical expression of how long a Geometric Brownian Motion process will stay below a given level averagely. The result is given in an inverse Laplace transform function, which significantly simplifies the computation.

Reference

BLACK, F. & SCHOLES, M. 1973. The pricing of options and corporate liabilities. *The journal of political economy***,** 637-654.

BOLLERSLEV, T., TAUCHEN, G. & ZHOU, H. 2009. Expected stock returns and variance risk premia. *Review of Financial studies,* 22**,** 4463-4492.

BRITTEN- JONES, M. & NEUBERGER, A. 2000. Option prices, implied price processes, and stochastic volatility. *The Journal of Finance,* 55**,** 839-866.

CARR, P. & WU, L. 2005. A Tale of Two Indices. *Journal of Derivatives,* 13.

CARR, P. & WU, L. 2009. Variance risk premiums. *Review of Financial Studies,* 22**,** 1311-1341.

CBOE, C. B. O. E. 2009. The CBOE volatility index–VIX. *White Paper*.

COVAL, J. D. & SHUMWAY, T. 2001. Expected Option Returns. *The Journal of Finance,* 56**,** 983-1009.

CREMERS, M., HALLING, M. & WEINBAUM, D. 2015. Aggregate Jump and Volatility Risk in the Cross-Section of Stock Returns. *The Journal of Finance,* 70**,** 577-614.

DEMETERFI, K., DERMAN, E., KAMAL, M. & ZOU, J. 1999. More than you ever wanted to know about volatility swaps. *Goldman Sachs quantitative strategies research notes,* 41.

JACKWERTH, J. C. 2000. Recovering Risk Aversion from Option Prices and Realized Returns. *The Review of Financial Studies,* 13**,** 433-451.

L VY, P. 1939. Sur certains processus stochastiques homogènes. *Comp Math***,** 283-339.

MADAN, D. B., CARR, P. P. & CHANG, E. C. 1998. The variance gamma process and option pricing. *European finance review,* 2**,** 79-105.

MERTON, R. C. 1976. Option pricing when underlying stock returns are discontinuous. *Journal of financial economics,* 3**,** 125-144.

PHAM, H. 2000. *Continuous-time stochastic control and optimization with financial applications*, Springer.

RYDBERG, T. H. 1997. The Normal Inverse Gaussian Lévy Process: Simulation and Approximation. *Communications in Statistics Part C Stochastic Models,* 13**,** 887-910.

SCHILLING, R. L. 2004. *Financial Modelling with Jump Processes*, Chapman & Hall/CRC.